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Concrete Mathematics - MA 201
Problem Sheet - 5

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1. What does the notation $\sum_{k=4}^{0} a_k$ mean?

2. Simplify the expression
$$x.([x > 0] - [x < 0]).$$

3. Demonstrate your understanding of $\sum$-notation by writing out the sums $\sum_{0 \le k \le 5} a_k$ and $\sum_{0 \le k^2 \le 5} a_{k^2}$ in full.

4. Express the triple sum $\sum_{1 \le i < j < k \le 4} a_{ijk}$ as a three-fold summation (with three $\sum$'s),
   (a) summing first on k, then j, then i ;
   (b) summing first on i, then j, then k.

   Also write your triple sums out in full without the $\sum$-notation using parentheses to show that is being added together first.

5. What is wrong with the following derivation?
$$\Big(\sum_{j=1}^{n} a_j\Big)\Big(\sum_{k=1}^{n} \frac{1}{a_k}\Big) = \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{a_j}{a_k} = \sum_{k=1}^{n}\sum_{k=1}^{n} \frac{a_k}{a_k} = \sum_{k=1}^{n} n = n^2.$$

6. What is the value of $\sum_k [1 \le j \le k \le n]$, as a function of j and n?

7. Let $\bigtriangledown f(x) = f(x) - f(x-1)$. What is $\bigtriangledown(x^{\overline{m}})$?

8. What is the value of $0^{\underline{m}}$, when m is a given integer?

9. Prove that the falling-power version of law of exponents:
$$x^{\underline{m+n}} = x^{\underline{m}}(x-m)x^{\underline{n}}, \quad \text{integers } m, n.$$
   What is the law of exponents for rising factorial powers, analogous to the above relation? Use this to define $x^{\overline{-n}}$.

10. We have proved the following formula for the difference of a product : $\Delta(uv) = u\Delta v + Ev\Delta u$. How can this formula be correct, when the left-hand side is symmetric with respect to u and v but the right-hand side is not?

11. The general rule
$$\sum u\Delta v = uv - \sum Ev\Delta u$$
   for summation by parts is equivalent to
$$\sum_{0 \le k < n} (a_{k+1} - a_k) b_k = a_n b_n - a_0 b_0 - \sum_{0 \le k < n} a_{k+1}(b_{k+1} - b_k), \quad \text{for } n \ge 0.$$
   Prove this formula directly by using the distributive, associative, and commutative laws.

12. Use the repertoire method to find a closed form for $\sum_{k=0}^{n}(-1)^k k^2$.

13. Evaluate $\sum_{k=1}^{n} k2^k$ by rewriting it as the multiple sum $\sum_{1\le j\le k\le n} 2^k$.

14. Evaluate $C_n = \sum_{k=1}^{n} k^3$ by "expand and contract method" as follows: First write

$$C_n + \square_n = 2\sum_{1\le j\le k\le n} jk$$

then apply

$$S_{\triangle} = \sum_{1\le j\le k\le n} a_j a_k = \frac{1}{2}\Big((\sum_{k=1}^{n} a_k)^2 + \sum_{k=1}^{n} a_k^2\Big).$$

15. Prove that $x^{\underline{m}}/(x-n)^{\underline{m}} = x^{\underline{n}}/(x-m)^{\underline{n}}$, unless one of the denominators is zero.

16. Show that the following formulas can be used to convert between rising and falling factorial powers, for all integers m:

$$x^{\overline{m}} = (-1)^m(-x)^{\underline{m}} = (x+m-1)^{\underline{m}} = 1/(x-1)^{\underline{-m}}$$

$$x^{\overline{m}} = (-1)^m(-x)^{\overline{m}} = (x-m+1)^{\overline{m}} = 1/(x+1)\overline{-m}.$$

17. Let $Re(z)$ and $Im(z)$ be the real and imaginary parts of the complex number z. The absolute value $|z|$ is $\sqrt{(Re(z))^2 + (Im(z))^2}$. A sum $\sum_{k\in K} a_k$ of complex terms a_k is said to converge absolutely when the real-valued sums $\sum_{k\in K} Re(a_k)$ and $\sum_{k\in K} Im(a_k)$ both converge absolutely. Prove that $\sum_{k\in K} a_k$ converges absolutely if and only if there is a bounding constant B such that $\sum_{k\in F} |a_k| \le B$ for all finite subsets $F \subseteq K$.

18. Use a summation factor to solve the recurrence

$$\begin{aligned} T_0 &= 5 \\ 2T_n &= nT_{n-1} + 3n!, \quad \text{for } n > 0. \end{aligned}$$

19. Try to evaluate $\sum_{k=0}^{n} kH_k$ by the perturbation method, but deduce the value of $\sum_{k=0}^{n} H_k$ instead.

20. Evaluate the sums

$$S_n = \sum_{k=0}^{n}(-1)^{n-k}, T_n = \sum_{k=0}^{n}(-1)^{n-k}k, \text{ and } U_n = \sum_{k=0}^{n}(-1)^{n-k}k^2$$

by the perturbation method, assuming that $n \ge 0$.

21. Prove **Lagrange's identity** (without using induction):

$$\sum_{1\le j<k\le n}(a_j b_k - a_k b_j)^2 = \Big(\sum_{k=1}^{n} a_k^2\Big)\Big(\sum_{k=1}^{n} b_k^2\Big) - \Big(\sum_{k=1}^{n} a_k b_k\Big)^2.$$

Prove, in fact, an identity for the more general double sum

$$\sum_{1\le j<k\le n}(a_j b_k - a_k b_j)(A_j B_k - A_k B_j).$$

22. Evaluate the sum $\sum_{k=1}^{n}(2k+1)/k(k+1)$ in two ways:

   (a) Replace $1/k(k+1)$ by the "partial fractions" $1/k - 1/(k+1)$.

   (b) Sum by parts.

23. What is $\sum_{0\le k<n} H_k/(k+1)(k+2)$?

24. The notation $\prod_{k\in K} a_k$ means the product of the numbers a_k for all $k \in K$.

    Assume for simplicity that $a_k \neq 1$ for only finitely many k; hence infinite products need not be defined.

    What law does this $\prod$-notation satisfy, analogous to the distributive, associative, and commutative laws that hold for $\sum$?

25. Express the double product $\prod_{1\le j\le k\le n} a_j a_k$ in terms of the single product $\prod_{k=1}^{n} a_k$ by manipulating $\prod$-notation.

26. Computer $\Delta(c^{\underline{x}})$, and use it to deduce the value of $\sum_{k=1}^{n}(-2)^{\underline{k}}/k$.

27. Evaluate the sum $\sum_{k=1}^{n}(-1)^k k/(4k^2-1)$.

28. Riemann's zeta function $\zeta(k)$ is defined to be the infinte sum

$$1 + \frac{1}{2^k} + \frac{1}{3^k} + \cdots = \sum_{j\ge 1}\frac{1}{j^k}.$$

    Prove that $\sum_{k\ge 2}(\zeta(k)-1) = 1$. What is the value of $\sum_{k\ge 1}(\zeta(2k)-1)$?

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